SURFACES OF GENERAL TYPE WITH $p_g = q = 0$ HAVING A PENCIL OF HYPERELLIPTIC CURVES OF GENUS 3

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Abstract. We prove that the bicanonical map of a surfaces of general type S with $p_g = q = 0$ is non birational if there exists a pencil $|F|$ on S whose general member is an hyperelliptic curve of genus 3.

Let S be a minimal surface of general type and let $f_g : S \dashrightarrow B$ be a rational map onto a smooth curve such that the normalization of the general fibre is an hyperelliptic curve of genus g. Then the hyperelliptic involution of the general fibre induces a (biregular) involution σ on S and one has a map of degree two $\rho : S \dashrightarrow \Sigma$ onto a smooth ruled surface with ruling induced by f_g.

It is known (e.g [2],[4]) that if $g = 2$ than the bicanonical map φ_{2K} of S factors through ρ and moreover, if $K_S^2 \geq 10$ then the non birationality of φ_{2K} implies the existence of f_g with $g = 2$ ([7]). On the other hand, we proved in [3] that if φ_{2K} factors through a rational map ρ (generically) of degree two onto a ruled surface, then there exists a map f_g where $g \leq 4$ and, more precisely, if $g \neq 2$ then we have $g = 3$ unless K_S is ample and $q(S) = 0, p_g(S) = \frac{1}{2}(d-3)d + 1$, $K_S^2 = 2(d-3)^2$, $d = 4, 5$. Finally, we recall that if $p_g(S) = 0$ and $K_S^2 \geq 3$, then there not exists f_g with $g = 2$ ([8]).

In this note we prove that if $p_g = 0$ then the existence of f_g with $g = 3$ implies φ_{2K} non birational. In particular, it follows that if $p_g = 0$ and $K_S^2 \geq 3$ then φ_{2K} factors through a map of degree two onto a ruled surface if and only if there exists f_g with $g = 3$.

To motivate this work we notice that, as far as we know, all the examples of surfaces of general type with $p_g = 0$ and non birational bicanonical map have an f_g with $g = 3$, except one case when $K_S^2 = 3$ and S is a double cover of an Enriques surface.

Notation and conventions. We work over the complex numbers. We denote by K_S a canonical divisor, by $p_g = h^0(S, \mathcal{O}_S(K_S)) = 0$ the geometric genus and by $q = h^1(S, \mathcal{O}_S(K_S)) = 0$ the irregularity of a smooth (projective algebraic) surface S. The symbol $\equiv$ (resp. $\sim$) will denote the linear (resp. numerical) equivalence of divisors. A curve on a surface has an $[r, r]$-point at p if it has a point of multiplicity r at p which resolves to a point of multiplicity r after one blow up.

1. SURFACES WITH A PENCIL OF CURVES OF GENUS 3

Assumption 1.1. *Throughout the end we assume that*

a) S is a minimal surface of general type with $p_g = q = 0$ and

b) $f : S \dashrightarrow \mathbb{P}^1$ is a rational map with connected fibres such that the normalization of general fibre F is a curve of genus $g = 3$.

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Proposition 1.2. *Let F be a general member of $|F|$. Then,*

- *either F has a double point and $F \sim 2K_S$, in this case $K_S^2 = 1$;*
- *or F is smooth and $F^2 = 2$, in this case $K_S^2 = 1$;*
- *or F is smooth and $F^2 \leq 1$.*

Proof. Suppose that F has multiplicity $m_i \geq 2$ at x_i and let $\overline{S} \to S$ be the blow up of S at the x_i's. Denote by $\overline{F} \subset \overline{S}$ the strict transform of F. Then, by the adjunction formula we have

$$4 - \overline{F}^2 - \sum m_i = 2g - 2 - \overline{F}^2 - \sum m_i = K_{\overline{S}}.\overline{F} - \sum m_i = K_S.F \geq 1$$

since K_S is nef and $h^0(F) > 0$. Hence, the Hodge Index Theorem says that

$$4 \geq (4 - \overline{F}^2 - \sum m_i)^2 = (K_S.F)^2 \geq K_S^2.F^2 \geq F^2 \geq \sum m_i^2 \geq 4$$

which implies that $F^2 = 4$ and F is numerically equivalent to $2K_S$. In particular, F has exactly a double point.

Analogously, if F is smooth we get that $F^2 \geq 2$ implies $F^2 = 2$ and $K_S^2 \leq 2$ with the equality only if F is numerically equivalent to K_S. In the latter case $F - K_S$ is a non zero torsion element $\mu \in Pic(S)$, and $h^0(F) - h^1(F) + h^2(F) = \chi(F) = \chi(K_S) = 1$ yields $h^1(-\mu) > 0$. Therefore, the unramified covering induced by μ is irregular. A contradiction, indeed $\pi_1^{alg}(S) \leq 9$ (cfr. [6]). □

2. The hyperelliptic case

Assumption 2.1. *From now on we assume that (the normalization of) the general member $F \in |F|$ is hyperelliptic.*

2.1. Involutions and double covers. Let σ_F be the hyperelliptic involution on F and denote by σ the involution induced on S. Then, σ is biregular since S is minimal of general type, and the fixed locus $Fix(\sigma)$ is union of a smooth curve R and finitely many isolated points $p_1, \ldots, p_\nu$.

Lemma 2.2. *The base points of $|F|$ belong to the fixed locus of σ; moreover if $F^2 < 4$ (i.e the general $F \in |F|$ is smooth), then they are distinct and isolated.*

Proof. Consider $\sigma(q_1)$ for a base point q_1 of $|F|$. If F, F' are general members of $|F|$ we have

$$\sigma_F(q_1) = \sigma(q_1) = \sigma_{F'}(q_1)$$

and so $\sigma(q_1) \in F \cap F'$. Therefore, $q_1 = \sigma(q_1)$ if $F^2 = 1$ or 4.

If $F^2 = 2$ denote by q_2 the other base point ($q_2 = q_1$ if F and F' are tangent). Then $\sigma(q_1) \in \{q_1, q_2\}$ and the exact sequence

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(F)) \to H^0(F, \mathcal{O}_F(q_1 + q_2)) \to 0$$

implies $h^0(F, \mathcal{O}_F(q_1 + q_2)) = 1$. Therefore, $q_1 = \sigma_F(q_1) \neq q_2$ and q_1, q_2 are fixed points of σ. In particular, F and F' meet transversally at the base points.

Suppose that there exists a component R_{q_i} of R passing through $q_i, i \in \{1, 2\}$, so that locally σ is the involution $(x, y) \mapsto (x, -y)$. Notice now that since F, F' meet transversally at q_i, they also meet transversally R_{q_i} at q_i. A contradiction, since F, F' are σ-invariant. □

Let $\hat{S} \to S$ be the blow up of S at $p_1, \ldots, p_\nu$ and denote by $\hat{\sigma}$ the induced involution, which is biregular since we are blowing up isolated fixed points. Let $\rho : \hat{S} \to \Sigma = \hat{S}/\hat{\sigma}$ be the projection onto the quotient. Hence, Σ is a smooth rational surface and ρ is a (finite) double cover. Denote by $\hat{B} = \rho(\hat{R}) = \rho(\pi^*(R) + \sum E_i)$ the branch curve of ρ, where $E_i = \pi^{-1}(p_i),\ i = 1, \ldots, \nu$. Then, $\hat{B}$ is a smooth curve linearly equivalent to $2\hat{\Delta}$ for some $\hat{\Delta} \in Pic(\hat{\Sigma})$. By [3, Proposition 1.2] we have the following equalities:

$$\nu = K_S.R + 4 = K_S^2 + 4 - 2h^0(\hat{\Sigma}, 2K_{\hat{\Sigma}} + \hat{\Delta}) \tag{2.1}$$

$$K_{\hat{S}}^2 = K_S^2 - \nu \tag{2.2}$$

and φ_{2K_S} factors trough ρ if and only if $h^0(\hat{\Sigma}, 2K_{\hat{\Sigma}} + \hat{\Delta}) = 0$. Notice that $\nu \geq 4$ since K_S is nef.

Lemma 2.3. *Assume $K_S^2 = 1$, then φ_{2K_S} factors trough ρ.*

Proof. Otherwise it would be $h^0(\hat{\Sigma}, 2K_{\hat{\Sigma}} + \hat{\Delta}) > 0$ and hence $\nu \leq 3$. □

From now on, in this section we assume $K_S^2 \geq 2$. Therefore, F is smooth and, by Lemma 2.2, Σ is birationally ruled by $|\hat{\Gamma}|$, where $\hat{\Gamma}$ is the image of F. Let $\omega : \Sigma \to \mathbb{F}_e,\ e \geq 0$ be a birational morphism such that $\Gamma = \omega_*(\hat{\Gamma})$ is a ruling, and consider a factorization $\omega = \omega_d \circ \cdots \circ \omega_1$ where $\omega_i : \Sigma_{i-1} \to \Sigma_i$ is the blow up at $q_i \in \Sigma_i, i \geq 1, \Sigma_0 := \Sigma$ and $\Sigma_d = \mathbb{F}_e$.

Denote by $\mathcal{E}_i, \mathcal{E}_i^*$ respectively the exceptional curve of ω_i and its total transform on Σ and by $B = \omega_*(\hat{B})$ denote the image of $\hat{B}$ on $\mathbb{F}_e$. Since $\hat{B} \equiv 2\hat{\Delta}$ we have $B \equiv 2\Delta$, where $\Delta = \omega_*(\hat{\Delta}) \in Pic(\mathbb{F}_e)$ and hence $B \equiv 8C_0 + 2b\Gamma$, where C_0 is the $(-e)$-section. Recall that $K_{\mathbb{F}_e} \equiv -2C_0 - (2+e)\Gamma$ and $K_\Sigma \equiv \omega^*(K_{\mathbb{F}_e}) + \sum \mathcal{E}_i^*$.

Finally, notice that the branch curve $\hat{B}$ contains exactly ν (-2)-curves (they correspond to the isolated fixed points of σ). In particular, the (-2)-curves arising from the base points of F map to sections of $\Sigma_e \to \mathbb{P}_1$, while each one of the others maps either to a point or to a fibre Γ.

Lemma 2.4. *In the above situation*

i) *$\hat{S}$ is the canonical resolution of the double cover of $\mathbb{F}_e$ branched along B;*

ii) *we can assume that:*

ii.a) *the essential singularities of B are at most $[5,5]$-points, and*

ii.b) *if $[x' \to x]$ is a $[5,5]$-point then the fibre Γ_x through x belongs to B;*

iii) *assume (ii), then*

iii.a) *there is a 1 to 1 correspondence*

$$\left\{ \begin{array}{c} (-2)\text{-curves contained} \\ \text{in } \hat{B} \text{ contracted to points} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} [r,r]\text{-points} \\ \text{of } B \end{array} \right\}$$

and

iii.b) *there is a 1 to 1 correspondence*

$$\left\{ \begin{array}{c} (-2)\text{-curves contained} \\ \text{in } \hat{B} \text{ which map to fibres} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{fibres passing through} \\ \text{a } [5,5]\text{-point of } B \end{array} \right\}$$

Proof. For $i)$ see [3, Lemma 1.3], while for $ii)$ and $iii)$ see [9, Lemmas 6 and 7]. □

Proposition 2.5. *Assume that ω has the property ii) of Lemma 2.4. Denote by a_1 (resp. a_2, a_3) the number of $[5,5]$-points (resp. $[3,3]$-points, 4 and 5-tuple points) of B. Then*

$$d \geq \ 2a_1 + \ 2a_2 + a_3; \quad \nu - F^2 = 2a_1 + a_2$$
$$\frac{3}{2}K_{\hat{S}}^2 + 12 = a_1 + a_2 + a_3$$

Proof. Since $\hat{B}$ is smooth, the inequality is clear, just notice that to resolve a singularity of type $[r, r]$ we need to blow up twice. The equality $\nu - F^2 = 2a_1 + a_2$ follows from $ii, b)$ and $iii)$ of the above lemma.

By Lemma 2.4 and [5], we have $\hat{B} = \omega^*(B) - \sum 2[\frac{m_i}{2}]\mathcal{E}_i^*$ where m_i is the multiplicity of B_i at q_i. Hence, since ρ is a double cover we get

$$\chi(\hat{S}) = \chi(\Sigma) + \chi(K_\Sigma + \hat{\Delta}) = \frac{1}{2}(K_{\mathbb{F}_e} + \Delta).\Delta + 2\chi(\mathbb{F}_e) - \frac{1}{2}\sum[\frac{m_i}{2}]([\frac{m_i}{2}] - 1)$$
$$K_{\hat{S}}^2 = 2(K_\Sigma + \hat{\Delta})^2 = 2(K_{\mathbb{F}_e} + \Delta)^2 - \sum([\frac{m_i}{2}] - 1)^2$$

and hence

$$\begin{aligned} 1 &= \frac{1}{2}(6b - 12e - 8) + 2 - \frac{1}{2}(a_1 8 + a_2 2 + a_3 2) = \\ &= 3b - 6e - 2 - 4a_1 - a_2 - a_3; \\ K_{\hat{S}}^2 &= 2(4b - 8e - 8) - 2(a_1 5 + a_2 1 + a_3 1) = \\ &= 8b - 16e - 16 - 10a_1 - 2a_2 - 2a_3 \end{aligned}$$

because B has at most $[5,5]$-points. Therefore,

$$3K_{\hat{S}}^2 - 8 = 3(8b - 16e - 16 - 10a_1 - 2a_2 - 2a_3) - 8(3b - 6e - 2 - 4a_1 - a_2 - a_3)$$

and so

$$\frac{3}{2}K_{\hat{S}}^2 + 12 = a_1 + a_2 + a_3$$

□

Remark 2.6. Because of the injection $H^0(\Sigma, \mathbb{C}) \hookrightarrow H^0(\hat{S}, \mathbb{C})$ induced by ρ we have $K_\Sigma^2 \geq K_{\hat{S}}^2$. Therefore, $d \leq 8 - K_\Sigma^2 \leq 8 - K_{\hat{S}}^2$ since ω is a sequence of d blow ups. We set $\hat{d} := 8 - K_{\hat{S}}^2$.

Lemma 2.7. *We have $[\frac{\nu - F^2}{2}] \geq a_1 \geq \nu - F^2 - \frac{1}{2}\hat{d}$*

Proof. The first inequality follows from Proposition 2.5. For the latter suppose that $a_1 = \nu - F^2 - \frac{1}{2}\hat{d} - i, i \geq 1$. Then we have $a_2 = \nu - F^2 - 2a_1 = \hat{d} + 2i - \nu + F^2$ and so $d \geq 2a_1 + 2a_2 = \hat{d} + 2i > \hat{d}$. A contradiction. □

Remark 2.8. Notice that if ω has the property $ii)$ of Lemma 2.4 and B has a $[5,5]$-point at p, then each irreducible component of B passing through p is tangent to the fibre which contains p. In particular, if $F^2 > 0$ then B contains F^2 sections of $\mathbb{F}_e \to \mathbb{P}_1$ and hence it has no $[5,5]$-points, i.e $a_1 = 0$.

3. THE MAIN RESULTS

Theorem 3.1. *Let S be a surface of general type with $p_g(S) = q(S) = 0$. Suppose that there exists a rational map $S \dashrightarrow \mathbb{P}_1$ such that the normalization of the general fibre F is an hyperelliptic curve of genus 3. Then,*

i) the bicanonical map of S is composed with the hyperelliptic involution.
ii) $K_S^2 \le 8$, and $K_S^2 \le 3$ if $F^2 = 1$.

Proof. By Lemma 2.3 and Proposition 1.2 we can assume $F^2 \le 1$. By Proposition 2.5, we have

$$\frac{3}{2}K_{\hat{S}}^2 + 12 = a_1 + a_2 + a_3$$

and by the above lemma we can write $a_1 = \nu - F^2 - \frac{1}{2}\hat{d} + i$, where

$$0 \le i \le \left[\frac{\nu - F^2}{2}\right] - \nu + F^2 + \frac{1}{2}\hat{d} \le \frac{1}{2}(\hat{d} - \nu + F^2)$$

Then, $a_2 = \nu - F^2 - 2a_1 = \hat{d} - \nu + F^2 - 2i$ and $a_3 \le \hat{d} - 2a_1 - 2a_2 = 2i$. Therefore, we get

$$\begin{aligned} 3K_{\hat{S}}^2 + 24 = 2\left(\frac{\hat{d}}{2} - i + a_3\right) &\le \hat{d} + 2i \le \\ &\le 2\hat{d} - \nu + F^2 = 16 - 2K_{\hat{S}}^2 - \nu + F^2 \end{aligned}$$

and hence,

$$4 \le \nu \le -5K_{\hat{S}}^2 - 8 + F^2 = 12 + F^2 - 10h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + \hat{\Delta})) \le 13 - 10h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + \hat{\Delta}))$$

by 2.1 and 2.2. It follows that $h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + \hat{\Delta})) = 0$ and whence φ_{2K} factors through the involution.

Finally, notice that:

a) if $F^2 = 0$ then $K_S^2 + 4 = \nu \le 12$, i.e. $K_S^2 \le 8$;
b) if $F^2 = 1$ then $a_1 = 0$, hence

$$12 = 2a_1 + 2a_2 + 2a_3 = 2(\nu - 1) + 2a_3 = 2K_S^2 + 6 + 2a_3$$

and so

$$K_S^2 = 3 - a_3 \le 3$$

In particular, $K_S^2 = 9$ does not occur. □

An involution ι on S is *rational* if the quotient S/ι is a rational surface. In the situation of Section 2 we have the following commutative diagram

$$\begin{array}{ccc} \hat{S} & \longrightarrow & S \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ \hat{\Sigma} & \longrightarrow & S/\sigma \end{array}$$

where S/σ is rational and, in particular, φ_{2K} factors through σ if and only if factors through ρ.

Theorem 3.2. *Let S be a minimal surface of general type with $p_g = q = 0$. Then the bicanonical map of S factors through a rational involution if and only if there exists a map $f_g : S \dashrightarrow \mathbb{P}_1$such that the normalization of the general fibre is an hyperelliptic curve of genus $g \leq 3$ ($g = 3$ if $K_S^2 \geq 3$).*

Proof. Assume that there exists f_g. If $g = 2$ see [2], [4] (in fact, the bicanonical system cuts on the general fibre a subserie of the g_2^1). If $g = 3$ then we have Theorem 3.1. Conversely, if the bicanonical map factors through a rational involution then by [3] there exists f_g with $g \leq 3$. Finally, Xiao G. [9, Theoreme 2] proved that if there exists f_g with $g = 2$ then $K_S^2 \leq 2$.

□

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